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Buckling of Sandwich Plates with Random Material Properties Using Improved Plate Model

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An improved higher-order zigzag plate theory in a random environment is proposed and it is implemented in a stochastic finite element framework to study the buckling characteristic of sandwich plates with random material properties. The theory satisfies transverse shear stress continuity at all of the layer interfaces and the transverse shear-stress-free condition at the top and bottom surfaces. It also includes the effect of transverse deformability to the core. The through-thickness variation of in-plane displacements is assumed to be cubic, whereas transverse displacement varies quadratically within the core and it remains constant over the face sheets. Note that sandwich and composite structures are characterized by inherent uncertainties in their material properties. The effect of these uncertainties on the buckling characteristic of sandwich plates is studied by modeling the macromechanical material properties as basic random variables. A stochastic finite element method consisting of an efficient C^0 finite element in conjunction with a mean-centered first-order perturbation approach is developed, and the model is employed to evaluate the second-order statistics of the buckling loads. The published results are used to validate the deterministic part of the proposed approach, and its stochastic component is validated with an independent Monte Carlo simulation.

Nomenclature

a	=	length of the plate
b	=	breadth of the plate
E	=	Young's modulus
G	=	shear modulus
h	=	total thickness of the plate
h_c, h_f	=	thicknesses of the core and face sheets
n_u, n_l	=	number of upper and lower layers
U_1, U_2	=	in-plane displacements of any point along the x and y directions
U_3	=	transverse displacements of any point along the z direction
u, v	=	in-plane displacements of any point at the midplane along the x and y directions
w	=	transverse displacements of any point at the midplane
w_u, w_l	=	transverse displacement at the top and the bottom layers of the core
α^i_{xu}, α^i_{yu}	=	change of slopes at the upper ith interface between the ith and $(i+1)$th layer
α^i_{xl}, α^i_{yl}	=	change of slopes at the lower ith interface between the ith and $(i+1)$th layer
$\{\delta\}$	=	element displacement vector
$\{\varepsilon\}$	=	strain vector at the midplane of the plate
$\{\bar{\varepsilon}\}$	=	strain vector at any point of the plate
θ_{lx}, θ_{ly}	=	rotation of the normal to the bottom face sheet and core interface about the y and x axes
θ_{ux}, θ_{uy}	=	rotation of the normal to the top face sheet and core interface about the y and x axes
θ_x, θ_y	=	rotation of the normal to the midplane about the y axis and x axis
$\{\bar{\sigma}\}$	=	stress vector at any point of the plate

I. Introduction

SANDWICH construction is a special type of laminated construction comprising stiff and strong face sheets separated by a soft and lightweight core material. It helps to achieve a significantly high stiffness/strength-to-weight ratio of sandwich plates. The sandwich plate structures are becoming steadily more popular in various engineering applications such as aerospace, marine, automotive structures, and similar weight-sensitive structures. In practical situations, these structures are often subjected to various types of in-plane loadings, which require a clear understanding of their buckling characteristics associated with the interaction between the face sheets and the core. In addition to the preceding, the sandwich plates show inherent randomness in their system properties due to many structural parameters associated with their complex manufacturing and/or fabrication processes, which cannot be strictly controlled. Thus, it is very important to assess the effect of the randomness in the system parameters on the structural response in an accurate and efficient manner. For this purpose, a probabilistic analysis is necessary, as it could not be achieved through a conventional deterministic analysis.

In the analysis of laminated sandwich/composite structures, transverse shear deformation plays a major role, due to the low shear modulus compared with extensional rigidity as well as significant variation of material properties between the layers. A large number of plate theories exist in literature to model the transverse shear deformation accurately and efficiently. Comprehensive reviews of the literature on the development of different plate theories have been carried out by many researchers [1,2]. The equivalent single-layer theories [3–5], in which the laminate is modeled as an equivalent single anisotropic layer, are unable to account for the transverse shear strain discontinuities at the layer interfaces. These discontinuities are much more severe in the sandwich plates, due to a wide variation of the material properties of the core and face sheets. To overcome the

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limitations of the equivalent single-layer theories, layerwise theories [6–9] have been proposed. These theories are quite accurate, as the variation of in-plane displacement along the plate thickness can be represented appropriately by taking unknowns at the different layer interfaces, which helps to incorporate the desired shear strain jump at the layer interfaces. The only problem with these theories is the number of unknowns, which becomes very large in a multilayered laminate, as the number of unknowns depends on the number of layers.

To retain the advantage of single-layer theories as well as layerwise theories, a new type of plate theory has been proposed [10–12] in which the beginning is made with a layerwise theory, but the unknowns at the different planes are expressed in terms of those at the reference plane through satisfaction of the transverse shear stress continuity at the layer interfaces. In this theory, the in-plane displacements have a piecewise linear variation across the plate thickness, giving a zigzag pattern. A further improvement is due to Di Sciuva [13], Bhaskar and Varadan [14], and Cho and Parmerter [15], who considered a cubic through-thickness variation of in-plane displacements with kinks at the layer interfaces. It also satisfies the transverse shear-stress-free condition at the plate top and bottom surfaces.

The refined plate theories [13–15] discussed in the previous paragraph perform well for the analysis of the sandwich plates with a core that is reasonably stiff in the transverse direction. But they are not adequate enough to accurately predict the behavior of the sandwich plates with a core such as polymer foam or low-strength nonmetallic honeycomb, which have a good degree of deformability in the transverse direction. These cores are often defined as *soft core*, for which the effect of transverse normal strain is found to be quite important. In fact, the transverse deformability of a low-strength core has some effect on the overall behavior of sandwich panels; furthermore, it is found to be crucial for the sandwich structures with localized effects such as concentrated loads [16]. However, apart from the conventional modeling techniques [17,18], a considerable amount of literature [19–24] is available on the buckling analysis of the sandwich plate without taking into account the effect of transverse normal deformation for the core, whereas limited studies are available on the analysis of the sandwich plate considering the effect of transverse normal deformation for the core [25–30]. So there is a need for proper accounting of the transverse normal deformation of the core to accurately predict the behavior of the sandwich plates with any type of core.

As discussed earlier, real-life structures always have some inherent randomness in their system parameters. Uncertainties are found in various structural parameters (e.g., material properties, geometry, etc.) due to complex manufacturing and/or fabrication processes, which result in variations in the values of these parameters, and a small change in their values may have a pronounced effect on the overall response of the structure. For a realistic prediction of the response of the sandwich laminates, the sensitivity of the material properties should be one of the most important studies and it should be properly investigated. Some efforts in this direction have been made by different researchers to incorporate the material uncertainty into the analysis of the conventional structures. Chen et al. [31] used the first-order perturbation technique to evaluate the effect of the uncertainty in geometrical and material properties for truss and beam problems. Shinozuka and Astill [32] obtained the statistical parameters of buckling loads using Monte Carlo simulation (MCS). Zhang and Ellingwood [33] used the second-order perturbation technique in conjunction with the stochastic finite element method to evaluate the effect of uncertain material properties on the stability of beams and frames.

Literature containing the analysis of sandwich/composite structure with random material properties is rather scanty. Leissa and Martin [34] studied the free-vibration and buckling characteristics of rectangular composite plates with variable fiber spacing using classical laminate theory. Nakagiri et al. [35] studied the eigenvalue problem of simply supported graphite/epoxy plates by taking fiber orientation, layer thickness, and layer numbers as random variables and found that the overall stiffness of fiber-reinforced composite laminated plates is largely dependent on the fiber orientation. Salim et al. [36] employed the first-order perturbation technique to analyze the buckling of composite plates with random material properties using classical laminate theory. They obtained the second-order statistical parameters of the buckling loads. Onkar et al. [37] presented a stochastic finite element formulation based on the mean-centered first-order perturbation technique for the buckling of laminated composite plates. They determined the statistical parameters of buckling strength by considering uncertainties in the material properties, assuming small dispersion about their mean values. In this direction, Singh et al. [38] used the first-order perturbation technique to study the effect of the random material properties on the buckling response of composite plates using a higher-order shear deformation theory. But the transverse shear stress continuity conditions at the layer interfaces are absent in their model. To the best of the authors' knowledge, there is no such effort to study the buckling characteristics of the laminated sandwich plates with random material properties addressing all of the aforementioned criteria.

Keeping all of these aspects in mind, an attempt has been made in this study to develop an improved plate model to predict the buckling characteristics of the laminated sandwich plate in the framework of the randomness in the material properties considering the effect of the transverse normal deformation of the core. The in-plane displacement fields are assumed to be a combination of a linear zigzag model with different slopes in each layer and a cubically varying function over the entire plate thickness. The out-of-plane displacement is assumed to vary quadratically within the core and to vary constantly throughout the faces. The plate model is implemented with a computationally efficient C^0 finite element formulation developed for this purpose and applied to solve a number of sandwich plate problems. An effort is also made to obtain second-order statistical parameters of the buckling loads of the sandwich plate using the stochastic finite element method (i.e., finite element method in conjunction with a mean-centered first-order perturbation technique). The material properties are considered to be probabilistic and are modeled as basic random variables, and the other system parameters are assumed to be deterministic in nature. The dispersions in the material properties are assumed to be small with respect to their mean values, which is the usual case in most of the engineering applications. The validation of the stochastic finite element approach is done by comparing the results with an independent Monte Carlo simulation.

II. Mathematical Formulation

The displacement vector at any point may be defined as

$$\mathbf{U} = \{U_i\} = \{\, U_1 \quad U_2 \quad U_3 \,\} \tag{1}$$

The through-thickness variation of in-plane displacements (Fig. 1) is assumed as follows:

$$U_1 = u + z\theta_x + \sum_{i=1}^{n_u-1}(z - z_i^u)H(z - z_i^u)\alpha_{xu}^i + \sum_{j=1}^{n_l-1}(z - z_j^l)H(-z + z_j^l)\alpha_{xl}^j + \beta_x z^2 + \eta_x z^3 \tag{2}$$

$$U_2 = v + z\theta_y + \sum_{i=1}^{n_u-1}(z - z_i^u)H(z - z_i^u)\alpha_{yu}^i + \sum_{j=1}^{n_l-1}(z - z_j^l)H(-z + z_j^l)\alpha_{yl}^j + \beta_y z^2 + \eta_y z^3 \tag{3}$$

where β_x, β_y, η_x, and η_y are the higher-order unknowns, and $H(z - z_i^u)$ and $H(-z + z_j^l)$ are the unit step functions.

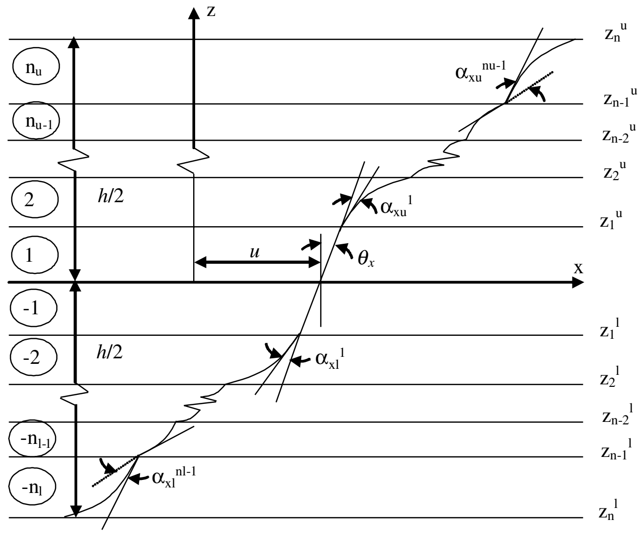

Fig. 1 General lamination scheme and displacement configuration.

where the indices i and j run over the sequences 1, 2, and 3 (unless indicated otherwise), and 1, 2, and 3 denote components along the x, y, and z directions, respectively, for a tensor quantity.

The constitutive relationship of an orthotropic layer/lamina (say, pth layer) with any fiber orientation with respect to structural axes system (x–y–z) may be expressed as

$$\bar{\sigma}_{ij} = \bar{C}^p_{ijkl}\bar{\varepsilon}_{kl} \tag{6}$$

where $\bar{C}^p_{ijkl}$ is the transformed constitutive tensor of the pth lamina [39].

Using the condition of zero transverse shear stress or zero transverse shear strain at the top and bottom surfaces of the plate and imposing the condition of the transverse shear stress continuity at the interfaces between the layers, the unknowns β_x, β_y, η_x, η_y, α^i_{xu}, α^i_{xl}, α^i_{yu}, and α^i_{yl}, as defined in Eqs. (2) and (3), may be expressed as

$$\boldsymbol{\beta} = \mathbf{A}\boldsymbol{\alpha} \tag{7}$$

where

$$\boldsymbol{\beta} = \{\beta_x \quad \eta_x \quad \beta_y \quad \eta_y \quad \alpha^1_{xu} \quad \alpha^2_{xu} \quad \cdots \quad \alpha^{n_u-1}_{xu} \quad \alpha^1_{xl} \quad \alpha^2_{xl} \quad \cdots \quad \alpha^{n_l-1}_{xl} \quad \alpha^1_{yu} \quad \alpha^2_{yu} \quad \cdots \quad \alpha^{n_u-1}_{yu} \quad \alpha^1_{yl} \quad \alpha^2_{yl} \quad \cdots \quad \alpha^{n_l-1}_{yl}\}^T$$

$$\boldsymbol{\alpha} = \{u \quad v \quad w \quad \theta_x \quad \theta_y \quad w_{u,x} \quad w_{u,y} \quad w_{l,x} \quad w_{l,y} \quad w_u \quad w_l\}^T$$

and the elements of $\mathbf{A}$ are functions of material properties.

The transverse displacement is proposed to vary quadratically over the core thickness, whereas it is constant over the upper and lower face sheets, as given in Fig. 2, and it may be expressed as follows:

For the core,

$$U_3 = l_1 w_u + l_2 w + l_3 w_l \tag{4a}$$

For upper faces,

$$U_3 = w_u \tag{4b}$$

For lower faces,

$$U_3 = w_l \tag{4c}$$

where l_1, l_2, and l_3 are Lagrangian interpolation functions in the thickness coordinate.

With the preceding displacement components at any point, the strain tensor may be expressed as

$$\bar{\varepsilon}_{ij} = \tfrac{1}{2}(U_{i,j} + U_{j,i}) \tag{5}$$

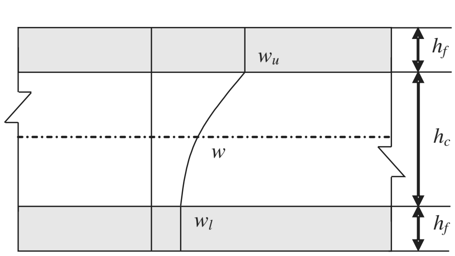

Fig. 2 Through-thickness variation of transverse displacement of a sandwich plate.

Using Eq. (7), the displacement field [Eqs. (2) and (3)] is modified and, accordingly, the strain tensor is obtained using Eq. (5). The strain tensor usually requires C^1 continuous shape functions for finite element approximation, as the modified displacement field contains first-order derivatives of w_u and w_l. To avoid the usual difficulties associated with satisfaction of C^1 continuity, $w_{u,x}$, $w_{u,y}$, $w_{l,x}$, and $w_{l,y}$ are substituted with θ_{ux}, θ_{uy}, θ_{lx}, and θ_{ly}, respectively, where θ_{ux}, θ_{uy}, θ_{lx}, and θ_{ly} are considered to be independent field variables. The preceding substitution imposes the following artificial constraints, which are enforced variationally through a penalty approach:

$$w_{u,x} - \theta_{ux} = 0, \quad w_{u,y} - \theta_{uy} = 0, \quad w_{l,x} - \theta_{lx} = 0, \quad w_{l,y} - \theta_{ly} = 0 \tag{8}$$

The strain tensor at any point $\bar{\boldsymbol{\varepsilon}}$ may be expressed in terms of strain vector $\boldsymbol{\varepsilon}$, defined in terms of unknowns at the reference planes $\boldsymbol{\alpha}$ as

$$\bar{\boldsymbol{\varepsilon}} = \mathbf{H}\boldsymbol{\varepsilon} \tag{9}$$

where $\boldsymbol{\varepsilon} = \{\boldsymbol{\alpha}^T \quad \boldsymbol{\alpha}^T_{,x} \quad \boldsymbol{\alpha}^T_{,y}\}^T$, and $\mathbf{H}$ is a function of z and unit step functions.

The buckling of a plate may be assumed to be a geometrically nonlinear problem for its accurate prediction. The generalized geometric strain vector for the present case is obtained using Green strains and may be expressed as

$$\boldsymbol{\varepsilon}_G = \begin{bmatrix} \frac{1}{2}U^2_{3,x} + \frac{1}{2}U^2_{1,x} + \frac{1}{2}U^2_{2,x} \\ \frac{1}{2}U^2_{3,y} + \frac{1}{2}U^2_{1,y} + \frac{1}{2}U^2_{2,y} \\ U_{3,x}U_{3,y} + U_{1,x}U_{1,y} + U_{2,x}U_{2,y} \end{bmatrix} = \frac{1}{2}\mathbf{A}_G\boldsymbol{\theta}_G \tag{10}$$

where

$$\mathbf{A}_G = \begin{bmatrix} U_{3,x} & 0 & U_{1,x} & 0 & U_{2,x} & 0 \\ 0 & U_{3,y} & 0 & U_{1,y} & 0 & U_{2,y} \\ U_{3,y} & U_{3,x} & U_{1,y} & U_{1,x} & U_{2,y} & U_{2,x} \end{bmatrix}$$

$$\boldsymbol{\theta}_G = [U_{3,x} \quad U_{3,y} \quad U_{1,x} \quad U_{1,y} \quad U_{2,x} \quad U_{2,y}]^T$$

and a comma indicates partial differentiation with respect to the coordinate subscripts that follow.

With the help of Eqs. (2–4), the preceding equation may be expressed as

$$\boldsymbol{\theta}_G = \mathbf{H}_G \boldsymbol{\varepsilon} \tag{11}$$

where $\mathbf{H}_G$ is also a function of z and unit step functions.

The strain energy of the plate system is expressed as

$$U_{\text{S.E.}} = \frac{1}{2}\sum_{p=1}^{N}\int_{\Omega} \bar{C}^p_{ijkl}\bar{\varepsilon}_{ij}\bar{\varepsilon}_{kl}\mathrm{d}\Omega = \frac{1}{2}\int_{\Delta} D_{mn}\varepsilon_m\varepsilon_n \mathrm{d}\Delta \tag{12}$$

where

$$D_{mn} = \sum_{p=1}^{N}\int \bar{C}^p_{ijkl}H_{ijm}H_{kln}\mathrm{d}z$$

and $m, n = 1, 2, \ldots, 33$.

The work done by applied in-plane or membrane forces is expressed as

$$W = \frac{1}{2}\int_{\Delta}[N_{xx}U^2_{3,x} + N_{yy}U^2_{3,y} + 2N_{xy}U_{3,x}U_{3,y}]\mathrm{d}\Delta \tag{13}$$

where N_{xx}, N_{yy}, and N_{xy} are the applied in-plane compressive forces.

In the present problem, the field variables based on the modified displacement field, as defined earlier, are u, v, w, θ_x, θ_y, θ_{ux}, θ_{uy}, θ_{lx}, θ_{ly}, w_u, and w_l. For finite element approximation of the in-plane variation of these field variables, a nine-node quadrilateral C^0 isoparametric element is used. The field-variable vector $\mathbf{f}$ at any point in the plane of an element may be expressed as

$$\mathbf{f} = \sum_{i=1}^{n} N_i \mathbf{f}^{(i)} = P_{lj}\delta_j \tag{14}$$

where $\mathbf{f}^{(i)}$ is the field-variable vector at the ith node; N_i is the corresponding shape function; $n = 9$ is the number of nodes of an element; P_{lj} consists of shape functions; $l = 1, 2, \ldots, 11$; and $j = 1, 2, \ldots, 99$. Note that $\mathbf{f}$ is simply equal to $\boldsymbol{\alpha}$ once $w_{u,x}$, $w_{u,y}$, $w_{l,x}$, and $w_{l,y}$ are substituted with θ_{ux}, θ_{uy}, θ_{lx}, and θ_{ly}, respectively, in $\boldsymbol{\alpha}$.

With the help of Eq. (14), the strain vector ε_i and the field-variable vector $\mathbf{f}$ or $\boldsymbol{\alpha}$ may be expressed in terms of nodal displacement vector as

$$\varepsilon_n = B_{nj}\delta_j, \qquad \alpha_l = f_l = P_{lj}\delta_j \tag{15}$$

where B_{nj} is the strain displacement matrix.

By using Eqs. (12) and (15), the strain energy can now be expressed as

$$U_{\text{S.E.}} = \frac{1}{2}\int_{\Delta} D_{mn}B_{mi}B_{nj}\delta_i\delta_j\mathrm{d}\Delta = \frac{1}{2}K_{ij}\delta_i\delta_j \tag{16}$$

where

$$K_{ij} = \int_{\Delta} D_{mn}B_{mi}B_{nj}\mathrm{d}\Delta \tag{17}$$

Similarly, using Eqs. (13) and (15), the work done by the applied in-plane or membrane forces can be written as

$$W = \frac{1}{2}\int_{\Delta} G_{mn}B_{mi}B_{nj}\delta_i\delta_j\mathrm{d}\Delta = \frac{1}{2}K_{G_{ij}}\delta_i\delta_j \tag{18}$$

where

$$K_{G_{ij}} = \int_{\Delta} G_{mn}B_{mi}B_{nj}\mathrm{d}\Delta, \qquad G_{mn} = \sum_{p=1}^{n}\int S^p_{ij}H_{G_{mi}}H_{G_{nj}}\mathrm{d}z \tag{19}$$

and the stress matrix S^p_{ij} may be expressed in terms of in-plane stress components of the pth layer as

$$[S^p] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 & 0 & 0 & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{xx} & \sigma_{xy} & 0 & 0 \\ 0 & 0 & \sigma_{xy} & \sigma_{yy} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{xx} & \sigma_{xy} \\ 0 & 0 & 0 & 0 & \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

The governing equation for buckling can be obtained by using the principle of stationary value of total potential energy, and it may written as follows:

$$\mathbf{K}\boldsymbol{\psi} - \lambda\mathbf{K_G}\boldsymbol{\psi} = \mathbf{0} \tag{20}$$

where λ is the critical load of buckling, and $\boldsymbol{\psi}$ is the global displacement vector defining the buckling mode shape.

In deterministic environments, the preceding equation can be solved by any conventional method. However, in random environments, a further mathematical exercise is needed to solve it for proper handling of the randomness in the material properties, which is given next.

Let us assume the set of $\mathbf{R}$ random (time-invariant) field variables

$$\mathbf{b}(x, y) = \{b_1(x, y)b_2(x, y)b_3(x, y)\cdots b_R(x, y)\} \tag{21}$$

where $b_i(x, y)$ represent random parameters (i.e., Poisson's ratio, Young's modulus, mass density, etc.).

Using finite element approximation, the random field variables may be expressed in terms of nodal random variables, similar to Eq. (14), as

$$\mathbf{b} = \sum_{i=1}^{n} N_i\mathbf{b}^{(i)} = F_{s\rho}b_\rho \tag{22}$$

where $\mathbf{b}^{(i)}$ is the field-variable vector at the ith node; $F_{s\rho}$ consists of shape functions; b_ρ are the nodal random field variables; $s = 1, 2, \ldots, R$; and $\rho = 1, 2, \ldots, 9R$.

For a structure with random parameters b_r, the individual terms of Eq. (20) will be random in nature, and thus it may be assumed that

$$K_{ij} = K_{ij}(b_\rho), \quad K_{G_{ij}} = K_{G_{ij}}(b_\rho), \quad \lambda_i = \lambda_i(b_\rho), \quad \psi_i = \psi_i(b_\rho) \tag{23}$$

where b_ρ is a vector of nodal random variables, as defined earlier.

Using the first-order perturbation approach [40], the random field variables b_r are expanded about their spatial expectations b^0_r with the help of a Taylor series with a given small parameter ξ and retained terms up to the first order. With this, the expansions of the terms of Eq. (20) may be expressed explicitly as

$$\begin{gathered} K_{ij} = K^0_{ij} + \xi K^{\rho}_{ij}\Delta b_\rho, \qquad K_{G_{ij}} = K^0_{G_{ij}} + \xi K^{\rho}_{G_{ij}}\Delta b_\rho \\ \lambda_i = \lambda^0_i + \xi\lambda^{,\rho}_i\Delta b_\rho, \qquad \psi_i = \psi^0_i + \xi\psi^{,\rho}_i\Delta b_\rho \end{gathered} \tag{24}$$

where $\xi\Delta b_\rho = \delta b_\rho = \xi[b_\rho - b^0_\rho]$ is the first-order variation of b_ρ about b^0_ρ. The superscript 0 represents the value of the functions taken at b^0_ρ, and superscript ", ρ" stands for the first partial derivative with respect to the nodal random field variables b_ρ evaluated at their expectations b^0_ρ.

Substituting Eq. (24) into Eq. (20), equating terms of like power of the small parameter ξ, the zeroth- and first-order equations for the stochastic structural buckling problem can be obtained as follows:

For zeroth order (ξ^0 terms, one system of n equations),

$$[K^0_{ij} - \lambda^0_k K^0_{G_{ij}}]\psi^0_{jk} = 0 \tag{25}$$

Table 1 Mean buckling-load parameters $\lambda_{cr} = N_x b^2/E_{22}h^3$ of a simply supported cross-ply square laminate under uniaxial compression N_x

	Thickness ratio h/a					
References[a]	0.01	0.02	0.05	0.1	0.25	0.5
Present (4 × 4)	35.9833	35.3596	31.5800	23.1347	8.1223	2.1176
Present (6 × 6)	35.9542	35.3315	31.5576	23.1226	8.1217	2.0327
Present (8 × 8)	35.9492	35.3267	31.5536	23.1203	8.1039	2.0112
Present (10 × 10)	35.9478	35.3254	31.5529	23.1199	8.0693	2.0061
Present (12 × 12)	35.9473	35.3250	31.5529	23.1199	8.0689	2.0060
Kant and Manjunatha [43]	35.9511	35.3409	31.6278	23.2527	8.8148	2.9495
Reddy [5]	35.9526	35.3467	31.6596	23.3400	8.9822	3.1514
Senthilnathan et al. [44]	36.0176	35.5981	32.9173	25.9651	10.6504	3.6802
Whitney and Pagano [45]	35.9550	35.3560	31.7071	23.4529	9.1138	3.1099

[a]Quantities within the parenthesis indicate mesh size.

For first order (ξ^1 terms, R systems of n equations),

$$[K^0_{ij} - \lambda^0_k K^0_{G_{ij}}]\psi^{,\rho}_{jk} = -[K^{,\rho}_{ij} - \lambda^{,\rho}_k K^0_{G_{ij}} - \lambda^0_k K^{,\rho}_{G_{ij}}]\psi^0_{jk} \tag{26}$$

Equation (25) is the equation for the eigenvalue problem in the deterministic sense. It can thus be solved for the zeroth-order eigenpairs λ^0_i and ψ^0_i by any conventional technique [41].

Knowing the mean eigenvalue and eigenvector, the first-order partial derivatives of eigenvalue and eigenvector with respect to the random variables are calculated to get their statistics. To do so, let us premultiply both sides of the first-order Eq. (26) by mean eigenvector ψ^{0T}_i to get

$$\psi^{0T}_{jk}[K^0_{ij} - \lambda^0_k K^0_{G_{ij}}]\psi^{,\rho}_{jk} - \psi^{0T}_{jk}\lambda^{,\rho}_k K^0_{G_{ij}}\psi^0_{jk} = -\psi^{0T}_{jk}[K^{,\rho}_{ij} - \lambda^0_k K^{,\rho}_{G_{ij}}]\psi^0_{jk} \tag{27}$$

Because λ^0_i is diagonal and K^0_{ij} and $K^0_{G_{ij}}$ are symmetric, the first term on the left-hand side of the preceding equation vanishes by the use of zeroth-order equation (25). Moreover, employing the orthonormality condition of $K^0_{G_{ij}}$, the expression for the first partial derivative of eigenvalues may be expressed as

$$\lambda^{,\rho}_k = \psi^{0T}_{jk}[K^{,\rho}_{ij} - \lambda^0_k K^{,\rho}_{G_{ij}}]\psi^0_{jk} \tag{28}$$

After getting the explicit expressions for the first-order derivative of the eigenvalues, the variance can be readily expressed as

$$\mathrm{var}\,(\lambda_i) = \sum_j^R \sum_k^R \lambda^{,j}_i \lambda^{,k}_i \mathrm{cov}(b_j, b_k), \qquad i = 1, 2, \ldots, 99 \tag{29}$$

where $\mathrm{cov}(b_j, b_k)$ is the covariance between b_j and b_k.

III. Results and Discussion

Various numerical examples are solved in this section to study the buckling response of laminated composite and sandwich plates with random material properties using the outlined probabilistic approach. The results obtained are compared with the published results and those obtained by an independent MCS to show the accuracy and performance of the present formulation. The second-order statistics (mean and standard deviation) of the buckling load are computed by varying the standard deviation (SD) to mean (μ) ratio of the material properties from 0 to 20% (Liu et al. [42]). The elastic properties of each lamina are modeled as the basic random variables b_i, which are denoted as

$$b_1 = E_{11}, \quad b_2 = E_{22}, \quad b_3 = E_{33}, \quad b_4 = \nu_{12}, \quad b_5 = \nu_{13}$$
$$b_6 = \nu_{23}, \quad b_7 = G_{12}, \quad b_8 = G_{13}, \quad b_9 = G_{23}$$

Table 2 Ratio of standard deviation and mean values ($SD_{\lambda_{cr}}/\mu_{\lambda_{cr}}$) of the nondimensional buckling load of a simply supported square cross-ply (0/90/90/0) laminated plate using Monte Carlo simulation

		$SD_{E_{11}}/\mu_{E_{11}}$			
h/a	Samples	0.05	0.10	0.15	0.20
0.01	2000	0.04497	0.08925	0.13362	0.18028
	4000	0.04507	0.08946	0.13678	0.18227
	6000	0.04558	0.09029	0.13583	0.18196
	8000	0.04530	0.09079	0.13704	0.18209
	10000	0.04530	0.09080	0.13705	0.18209
0.10	2000	0.02889	0.05757	0.08814	0.11833
	4000	0.02915	0.05787	0.08843	0.11894
	6000	0.02932	0.05830	0.08926	0.11913
	8000	0.02945	0.05866	0.08948	0.11988
	10000	0.02949	0.05870	0.08950	0.11990

A. Symmetric Cross-Ply Laminated Plate with Simply Supported Edges

A cross-ply (0/90/90/0) square laminated plate, simply supported at the four sides and subjected to uniaxial compressive load N_x, is considered. The analysis is performed for different thickness ratios: $h/a = 0.01$, 0.02, 0.05, 0.1, 0.25, and 0.5, where h and a are the thickness and sides of the square plate, respectively. The individual layer possesses the same thickness and material properties:

$$\frac{E_{11}}{E_{22}} = 40, \quad G_{12} = G_{13} = 0.6E_{22}, \quad G_{23} = 0.6E_{22}, \quad \nu_{12} = 0.25$$

It may be considered to be a three-layer laminated sandwich construction, wherein the thickness of the core is $h_c = h/2$ and that of each face is $h_f = h/4$. The mean buckling-load parameters $\lambda_{cr} =$

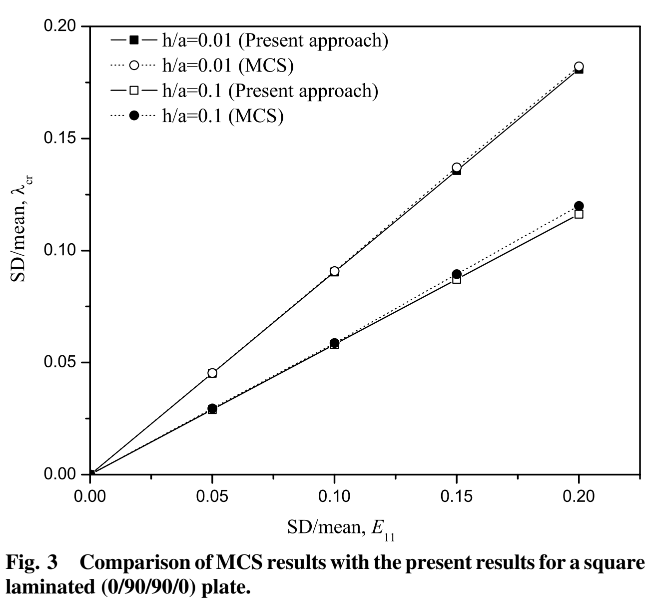

Fig. 3 Comparison of MCS results with the present results for a square laminated (0/90/90/0) plate.

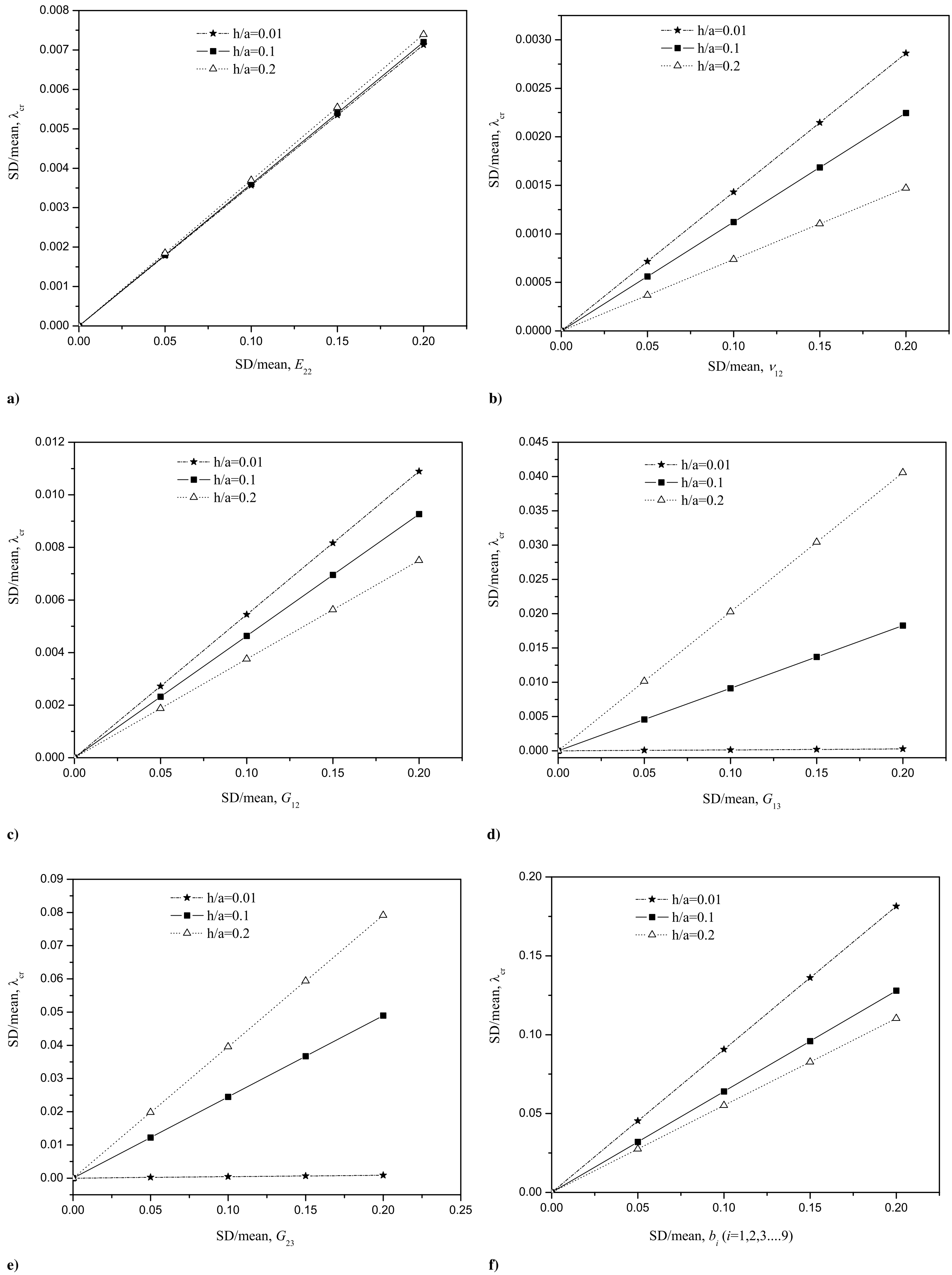

Fig. 4 Effect of scatter in material properties on the variation of nondimensional buckling load of a square laminated (0/90/90/0) plate: a) only E_{22} varying, b) only 12 varying, c) only G_{12} varying, d) only G_{13} varying, e) only G_{23} varying, and f) all material properties varying simultaneously.

Table 3 Mean buckling-load parameter $\lambda_{cr} = N_x b^2/E_{22f} h^3$ of a square sandwich (0/90/C/90/0) plate

a/h	Present	Dafedar et al. [27]	Kant and Manjunatha [43]	Reddy [5]	Senthilnathan et al. [44]	Whitney and Pagano [45]
2	0.0109	0.0109	0.0315	0.0583	0.0627	0.5995
4	0.0190	0.0190	0.0972	0.2115	0.2159	1.8339
10	0.0585	0.0585	0.5181	1.0909	1.0951	4.3197
20	0.1863	0.1862	1.6220	2.7913	2.7946	5.3628
30	0.3802	0.3801	2.6932	3.9213	3.9235	5.5983
40	0.6457	0.6457	3.5256	4.5695	4.5714	5.6899
50	0.9775	0.9775	4.1139	4.9634	4.9646	5.7495
60	1.3695	1.3694	4.5323	5.2100	5.2110	5.7855
70	1.8146	1.8145	4.8091	5.3553	5.3560	5.7907
80	2.2575	2.2569	5.0164	5.4610	5.4615	5.8013
90	2.7116	2.7109	5.1657	5.5328	5.5332	5.8053
100	3.1329	3.1324	5.2794	5.5862	5.5866	5.8097

$N_x b^2/E_{22} h^3$ obtained in the present analysis are given in Table 1. To show the convergence, the results have been computed with various mesh sizes. The present results are compared with those obtained from various shear deformation theories based on an equivalent single-layer model as reported by Kant and Swaminathan [26]. It is observed that the equivalent single-layer theories [5,43–45] overestimate the buckling loads when compared with the present plate model. The deviation of the present results from those of equivalent single-layer theories increases with the increase in the plate thickness ratio, as expected, due to the additional features adopted in the present model. The table clearly shows that the proposed theory can predict the buckling loads accurately with excellent convergence.

To show the influence of the randomness in the material properties on the buckling loads, the standard-deviation-to-mean ratio SD/μ of the basic random variables b_i is varied from 0 to 20% (Liu et al. [42]). To validate the present approach, the SD results are compared with those obtained from an independent MCS. The convergence of MCS results for critical buckling-load parameters corresponding to $h/a =$ 0.01 and 0.1 is presented by generating five different samples in Table 2, where E_{11} is taken as the only random variable. For MCS, the random numbers are generated assuming the material properties to have a Gaussian distribution. However, the present perturbation approach does not have any limitation on the distribution of the material properties, which is an advantage over the MCS. From the table, it is clear that 10,000 samples are sufficient to get the desired convergence in MCS. It is also noted that the variation in the buckling-load parameters reduces with the increase in the thickness ratio.

The scatter in the buckling-load parameters with the variation of SD/mean of E_{11} obtained from the present perturbation approach has been plotted with MCS in Fig. 3 for two different thickness ratios: $h/a = 0.01$ and 0.1. It is obvious from the figure that the present results are in excellent agreement with the MCS for both of the thickness ratios considered. Figures 4a–4e show the dispersion of the buckling-load parameters with the variation in the individual material property (E_{22}, ν_{12}, G_{12}, G_{13}, and G_{23}) for $h/a = 0.01$, 0.1, and 0.2, and Fig. 4f shows the variation of the buckling-load parameters when all of the properties are varying simultaneously. The effect of the variation of ν_{12} and G_{12} on the scatter in the nondimensional buckling loads decreases as the h/a ratio increases, whereas it increases with the individual variation in E_{22}, G_{13}, and G_{23} as the h/a ratio increases. For the simultaneous-variation case, the scatter in the buckling-load parameters decreases with the increase in the thickness ratio. It is also observed that E_{11} is the most dominating material property for the buckling of the laminates.

B. Simply Supported Sandwich Plate with Laminated Cross-Ply Face Sheets

A square symmetric sandwich plate (0/90/C/90/0), simply supported at the four sides and subjected to uniaxial compressive load N_x, is analyzed in this example. The ratio of thickness of the core h_c to the thickness of a face sheet h_f is taken as 10. Each ply constituting the face sheet has a thickness of $0.5h_f$. The mean values of the material properties of the core and face-sheet plies are given next:

Face sheets (orthotropic):

$$E_{11} = 131 \text{ GPa}, \qquad E_{22} = E_{33} = 10.34 \text{ GPa}$$
$$G_{12} = G_{23} = 6.895 \text{ GPa}, \qquad G_{13} = 6.205 \text{ GPa}$$
$$\nu_{12} = \nu_{13} = 0.22, \qquad \nu_{23} = 0.49$$

Core (isotropic):

$$E_{11} = E_{22} = E_{33} = 6.89 \times 10^{-3} \text{ GPa}$$
$$G_{12} = G_{23} = G_{13} = 3.45 \times 10^{-3} \text{ GPa}, \qquad \nu_{12} = \nu_{13} = \nu_{23} = 0$$

The variation of the mean critical buckling-load parameter $\lambda_{cr} = N_x b^2/E_{22f} h^3$ with respect to the side-to-thickness ratio a/h of the sandwich plate obtained from the proposed improved plate theory is presented in Table 3. It also includes the results obtained by Dafedar et al. [27] and those produced by Kant and Swaminathan [26] using different plate theories to compare the present results. The plate theories used by Kant and Swaminathan are the first-order shear deformation theory of Whitney and Pagano [45], higher-order (third-order) shear deformation theory of Reddy [5] and Senthilnathan et al. [44], and third-order shear deformation theory of Kant and Manjunatha [43] in which the effect of transverse normal deformation is considered. All of the plate theories adopted by Kant and Swaminathan [26] are based on equivalent single-layer theory. On the other hand, Dafedar et al. [27] presented two mixed higher-order models based on layerwise theory in one case and an equivalent single-layer theory in the other case.

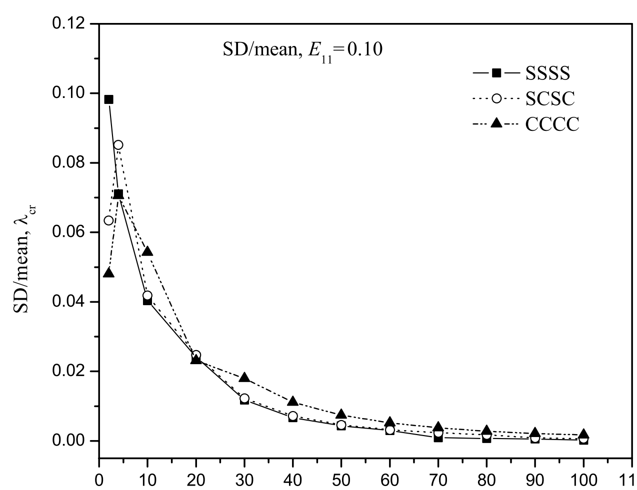

Fig. 5 Variation of SD/mean of nondimensional buckling load with the side-to-thickness ratio of a square sandwich (0/90/core/90/0) plate with different boundary conditions.

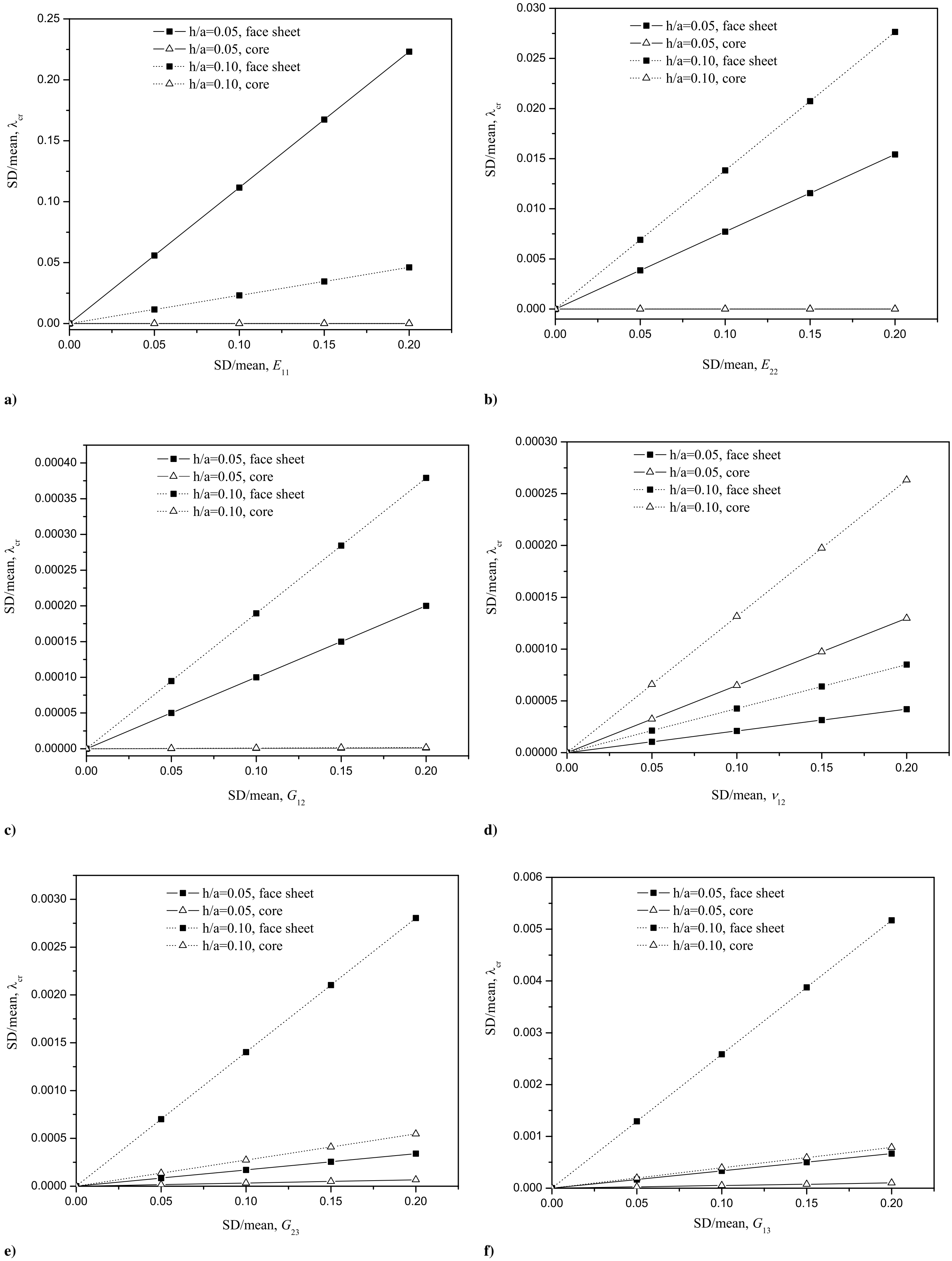

Fig. 6 Effect of scatter in material properties on the variation of nondimensional buckling load of a square sandwich (0/90/0/90/0/core/0/90/0/90/0) plate: a) only E_{11} varying b) only E_{22} varying, c) only G_{12} varying, d) only ν_{12} varying, e) only G_{23} varying, and f) only G_{13} varying.

The layerwise theory is found to be quite powerful when the unknowns consisting of transverse stress components in addition to the usual displacement components are taken at all of the layer interfaces, including the top and bottom surfaces of the plate. It involves a large number of unknowns hampering the computational efficiency severely, but the model definitely has the merit of providing results of high accuracy. It is observed that the values of buckling-load parameters obtained by layerwise theories are significantly lower than those obtained by the equivalent single-layer theories. The results also clearly show the performance of the different plate theories. It is interesting to note that the performance of the proposed model consisting of a reasonably lower number of unknowns is as good as that of the layerwise model [27]. Hence, the proposed model is found to be quite efficient in terms of the computational effort required for achieving that range of solution accuracy.

Now the plate is analyzed to assess the effect of the scatter in the material property on its buckling behavior. It is seen in the previous example that the most dominating material property for the buckling of the laminate is longitudinal Young's modulus E_{11}. So E_{11} is considered to be the only random variable in this problem, and the influence of having uncertain E_{11} on the buckling load of the sandwich plate is studied for different boundary conditions. The boundary conditions considered are simply supported at all four edges (SSSS), all edges clamped (CCCC), and two opposite edges simply supported with the other two clamped (SCSC).

The SD/mean of the critical buckling loads is calculated for each boundary condition with different thickness ratios and is presented in Fig. 5. Note that the SD/mean of E_{11} for the present example is taken to be 10%. It is observed that thicker plates are more sensitive to the variation in E_{11}, irrespective of the boundary conditions considered, and the SD/mean of the critical buckling load gradually decreases as the plate becomes thinner. The influence of scatter in E_{11} on the buckling loads is highest for the CCCC case, whereas it is least for the SSSS case for thin plates. But the highly thick plate with the SSSS condition shows the greatest variation of the buckling load.

C. Sandwich Plates with Laminated Face Sheets and Orthotropic Core

This example deals with a square sandwich plate (0/90/0/90/0/core/0/90/0/90/0), simply supported at the four sides and subjected to uniaxial compressive load N_x. Each laminated face sheet has a thickness of $h_f = 0.1h$ consisting of five equally thick plies, and the core has a thickness of $h_c = 0.8h$, where h is the total thickness of the plate. The mean values of the material properties are given next:

Face sheets (orthotropic):

$$\frac{E_{11}}{E_{22}} = 19, \qquad \frac{G_{12}}{E_{22}} = \frac{G_{13}}{E_{22}} = 0.52, \qquad \frac{G_{23}}{E_{22}} = 0.338$$

$$\nu_{12} = \nu_{13} = 0.32, \qquad \nu_{23} = 0.49$$

Core (orthotropic):

$$\frac{E_{11}}{E_{22f}} = 3.2 \times 10^{-5}, \qquad \frac{E_{22}}{E_{22f}} = \frac{G_{13}}{E_{22}} = 2.9 \times 10^{-5}$$

$$\frac{E_{33}}{E_{22f}} = 0.4, \qquad \frac{G_{12}}{E_{22f}} = 2.4 \times 10^{-3}, \qquad \frac{G_{13}}{E_{22f}} = 7.9 \times 10^{-2}$$

$$\frac{G_{23}}{E_{22f}} = 6.6 \times 10^{-2}, \qquad \nu_{12} = 0.99, \qquad \nu_{13} = \nu_{23} = 3 \times 10^{-5}$$

The mean values of the normalized critical buckling load $\lambda_{cr} = N_x b^2/E_{22f}h^3$ obtained by the outlined approach are presented with the 3-D elasticity solution (Noor et al. [29]) for two different thickness ratios ($h/a = 0.05$ and 0.1) in Table 4. The percentage difference between the results obtained from these two sources is also presented in Table 4, which indicates an excellent performance of the present improved higher-order displacement model.

Now the mean and standard deviations of the critical buckling load are evaluated to show the influence of the scattering in the material properties on the buckling response by allowing the SD/mean to vary from 0 to 20%. Figures 6a–6f show the variation of SD/mean of the nondimensional buckling loads with the variation of SD/mean of one material property at a time, treating the other material properties as deterministic. In general, it is observed that the scattering of the buckling load is the most sensitive to the longitudinal elastic modulus E_{11} for both of the thickness ratios ($h/a = 0.05$ and 0.1), whereas it is the least sensitive to changes in υ_{12} and the in-plane shear modulus. The effect of the core and the face-sheet properties on the scattering of the response is plotted separately for each thickness ratio, which clearly shows that the buckling loads are less sensitive to the changes in the core properties, compared with those of the face sheets, except ν_{12}. The effect of E_{11} on the scatter in a nondimensional buckling load decreases as the thickness ratio increases, whereas an opposite trend is seen in the case of E_{22}, ν_{12}, G_{12}, G_{13}, and G_{23} of the core as well as the face sheets. It is observed that the change in transverse elastic modulus E_{33} of the core has far less of an effect on the buckling response of the sandwich plate and is not presented here.

Table 4 Nondimensional mean buckling load $\lambda_{cr} = N_x b^2/E_{22f}h^3$ of a square symmetric sandwich (0/90/0/90/0/core/0/90/0/90/0) plate with laminated face sheets

	Thickness ratio h/a	
References	0.05	0.1
Present (4 × 4)	7.8713	5.5505
Present (6 × 6)	7.8658	5.5478
Present (8 × 8)	7.8649	5.5475
Present (10 × 10)	7.8646	5.5474
Present (12 × 12)	7.8646	5.5474
3-D elasticity (Noor et al. [29])	7.8969	5.6081
% error	0.41	1.09

The variation of the buckling-load parameters with simultaneous change in all of the material properties (i.e., all of the basic random variables) is shown in Figure 7. It is seen that the variation in buckling load reduces for a higher thickness ratio, and this effect is similar to the influence of individual variation in E_{11} of the face sheet. The simultaneous variation in core properties is not significant for both of the thickness ratios considered, whereas the changes in the face-sheet properties have a pronounced effect on the buckling response of the sandwich plate.

IV. Conclusions

In this study, an improved higher-order zigzag plate model is developed for the buckling analysis of a sandwich plate with random

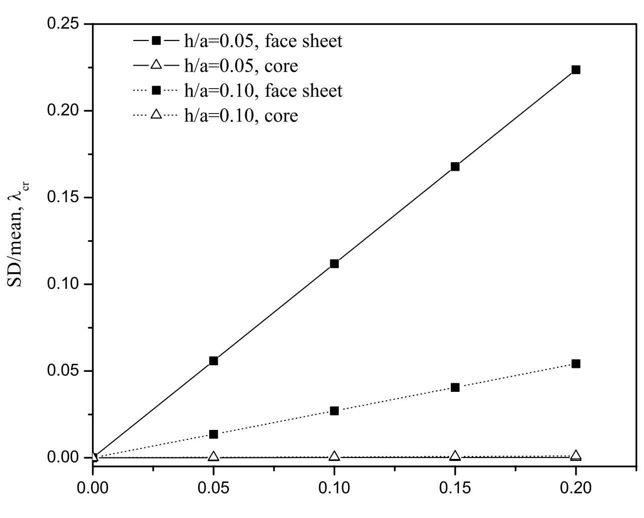

Fig. 7 Effect of scatter in material properties on the variation of nondimensional buckling load of a square sandwich (0/90/0/90/0/core/0/90/0/90/0) plate when all properties of the face sheets and the core are varying simultaneously.

material properties with a transversely flexible core. The most attractive feature of the plate theory is that it is capable of proper accounting of the transverse shear deformation as well as the transverse normal deformation of the flexible core with a sufficiently lower number of unknowns. An efficient C^0 finite element formulation in conjunction with a mean-centered first-order perturbation technique is developed. The resulting stochastic finite element model is validated by comparing the results obtained by the proposed approach with those obtained from an independent Monte Carlo simulation. It is observed that the scattering of the buckling-load parameters shows linear variation with standard deviation of the material properties. In general, the variations in longitudinal elastic modulus E_{11} of the face sheet have the most significant influence on the scattering of the buckling load of the sandwich plates, whereas it is the least sensitive to the changes in properties of the core material.

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